Definition: Average Rate of Change

An average rate of change is a division of two quantities representing the amount of change from one point to the next:

It is calculated using the standard slope formula: $\frac{y_2-y_1}{x_2-x_1}$

Note that we always choose y as the dependant variable and x as the independent variable.

Example 1:

Determine the average rate of change of a car travelling an 800 km trip from 1 pm to 11pm.

Solution:

The independent variable is time, the dependent variable is distance (As the total distance traveled depends on how long you are traveling). Our two points are $(1,0)$ and $(11, 800)$.

$$
\begin{aligned}
\text{AROC} \quad &= \frac{y_2-y_1}{x_2-x_1} \\
&= \frac{800-0}{11-1} \\
&= \frac{800}{10} \\
&= 80 \text{ km/h}
\end{aligned}
$$

Examples: Average Rate of Change

Example 2:

Given a function $f(x) = 3x^2 - \sin(x)$

Determine the average rate of change from $x = 0$ to $x = \pi$

Solution:

We first note that $f(0) = 3(0)^2 - \sin(0) = 0$ and $f(\pi) = 3\pi^2 - \sin(\pi) = 3\pi^2$

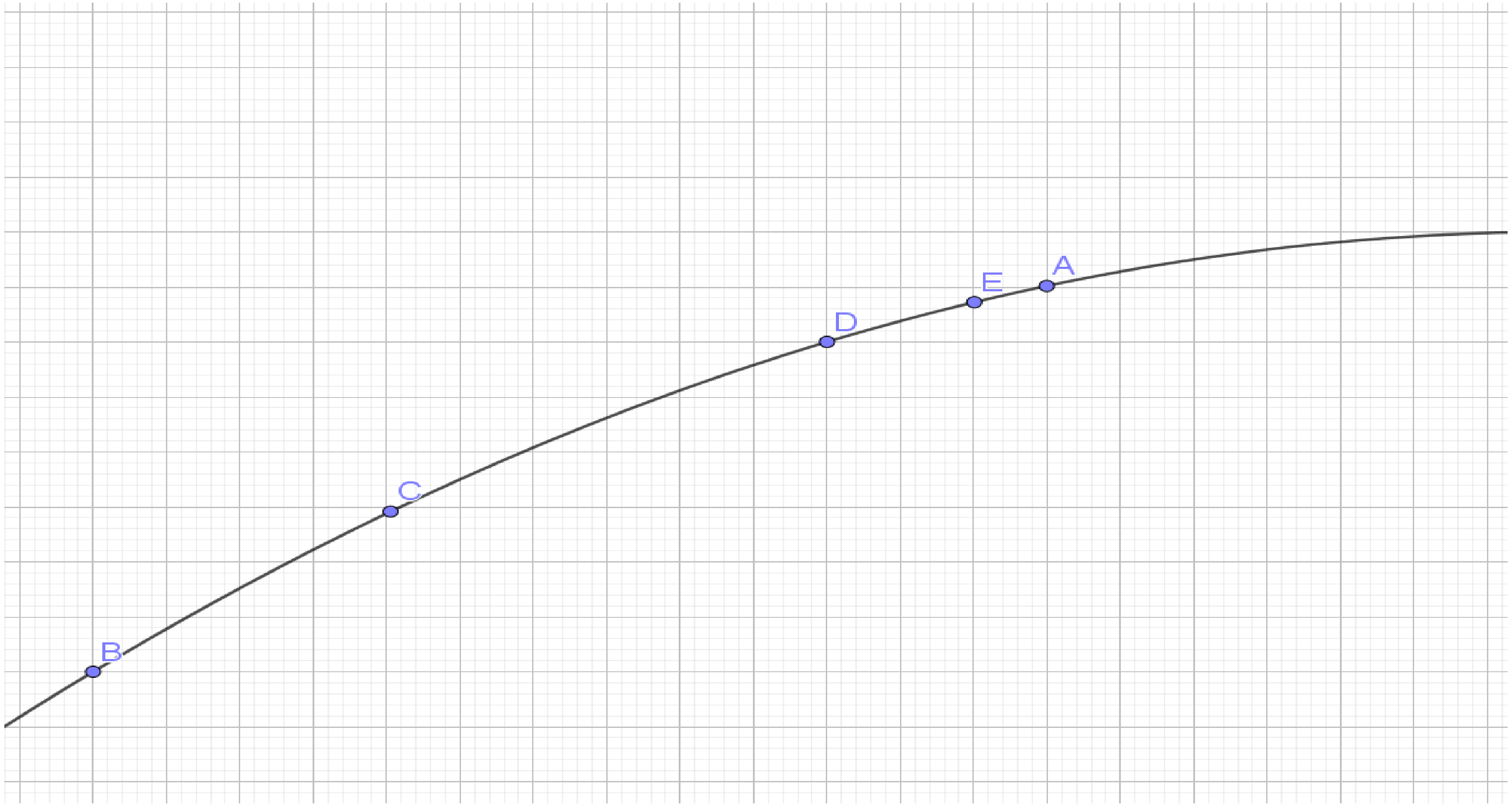
$\therefore$ Our two points are $(0,0)$ and $(\pi, 3\pi^2)$

$$\begin{aligned}
\text{AROC} \quad &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{3\pi^2 - 0}{\pi - 0} \\
&= \frac{3\pi^2}{\pi} \\
&= 3\pi
\end{aligned}$$

Definition: Instantaneous Rate of Change

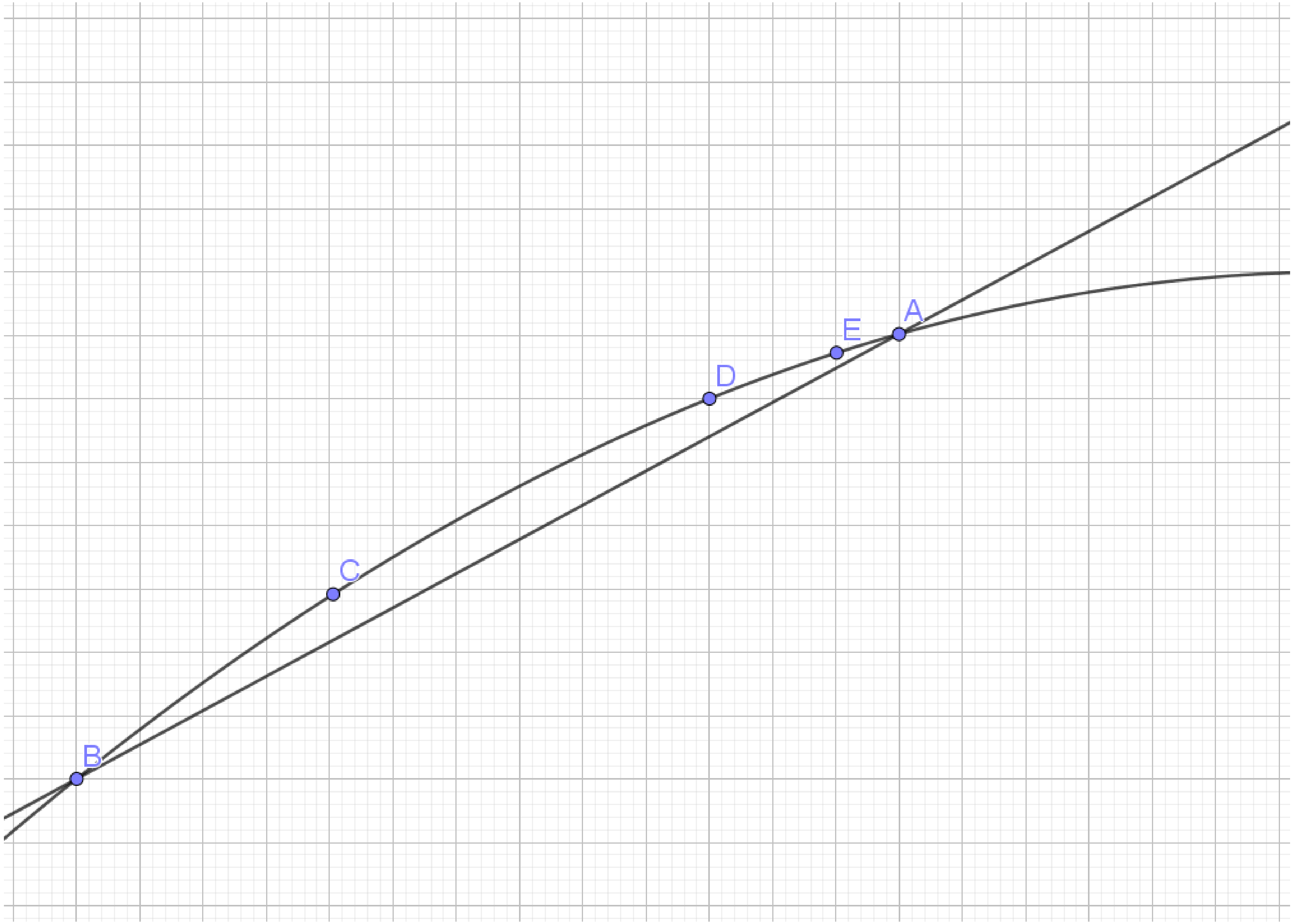
Given a function and a point $x = a$, we say that the instantaneous rate of change is the slope of the function at that point (also known as the tangent slope). Since we need two points to calculate slope, we actually need the idea of a limit to evaluate the instantaneous slope:

Given the picture below, if we wanted to find the instantaneous slope at point A:



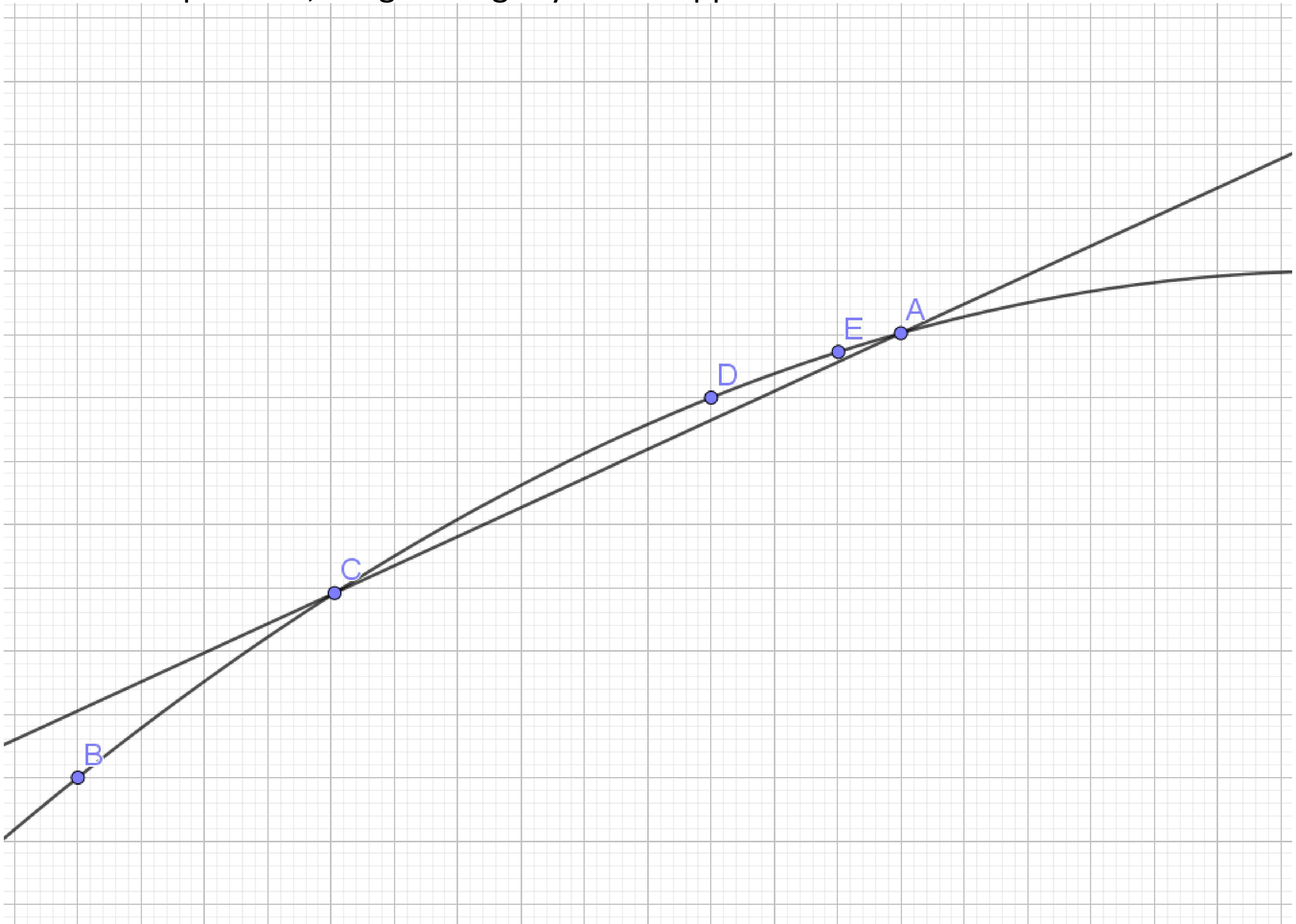
Definition: Instantaneous Rate of Change

We can use other points on the graph to find the slope. If we start with a slope AB we would get one approximation:



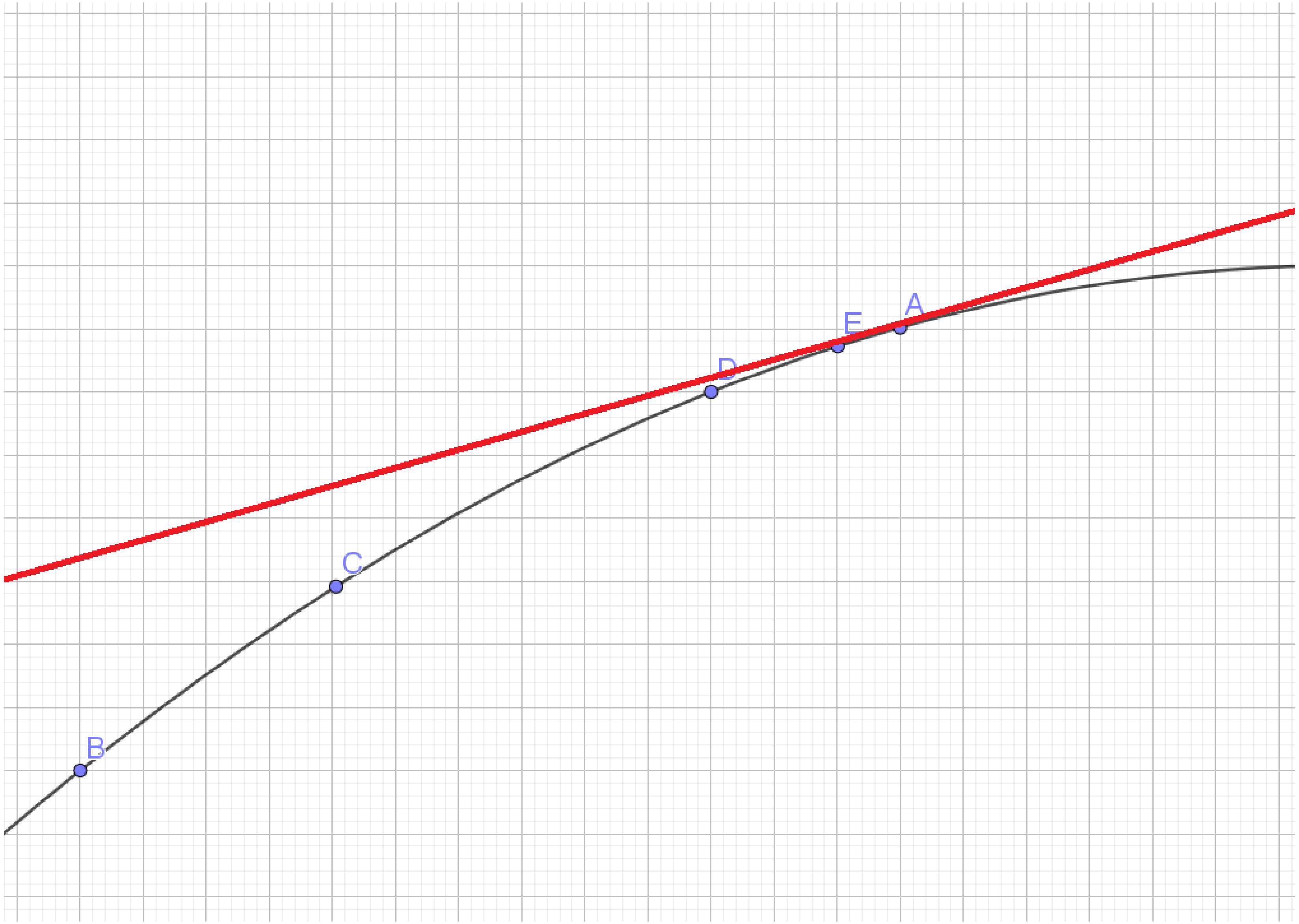
Definition: Instantaneous Rate of Change

If we instead find the slope of AC, we get a slightly better approximation:



Definition: Instantaneous Rate of Change

We can continue to do this (finding slope AD, then AE, then ….) until we find the instantaneous slope.



Definition: Instantaneous Rate of Change

We can calculate this using the following limit formula to find the instantaneous rate of change at $x = a$

IROC = $\lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}$

Example 3:

Determine the instantaneous rate of change at $x = 3$ on the function $f(x) = x^2 - 4x$:

Solution:

We use the formula to calculate the IROC: $a = 3$ $f(x) = x^2 - 4x, f(a) = f(3) = -3$

IROC $= \lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}$

$= \lim\limits_{x \to 3} \frac{f(x)-f(3)}{x-3}$

$= \lim\limits_{x \to 3} \frac{x^2-4x-[-3]}{x-3}$

$= \lim\limits_{x \to 3} \frac{x^2-4x+3}{x-3}$

$= \lim\limits_{x \to 3} \frac{(x-3)(x-1)}{x-3}$

$= \lim\limits_{x \to 3} x - 1$

$= 3 - 1$

$=2$

Examples: Instantaneous Rate of Change

Example 4:

Determine the equation of the tangent line of $f(x) = \sin(x)$ at $x = 0$

Solution:

To find the equation of the tangent line, we need the slope of the tangent (which is the IROC) at $a = 0, f(x) = \sin(x),\ f(0) = 0$

IROC $= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

$= \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$

$= \lim_{x \to 0} \frac{\sin(x) - 0}{x}$

$= \lim_{x \to 0} \frac{\sin(x)}{x}$

$= 1$ (identity limit)

Next we can find the equation of the line by using $m = 1$ and the point $(0,0)$ that is on the line:

$y = mx + b$

$0 = 1(0) + b$

$0 = b$

$\therefore$ The equation of the tangent line at $x = 0$ is $y = x$

Examples: Instantaneous Rate of Change

Example 5:

Suppose we are given a limit $\lim_{x\to -1} \frac{x^{2018}-1}{x+1}$. What function and at what point would this be finding an instantaneous rate of change?

Solution:

Here we see:

$$\text{IROC} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{x\to -1} \frac{x^{2018}-1}{x+1}$$

If we match up the numbers, we hopefully see that $a = -1$, and $f(x) = x^{2018}$. We should also (just in case) check that $f(a) = f(-1) = (-1)^{2018} = 1$ which also matches in our limit along with $x - a = x - (-1) = x + 1$.

Definition: Increasing and Decreasing

We say a function is increasing if the graph moves up and to the right. We say a function is decreasing if the graph is moving down and to the right.

We note that there is a clear indication if a function is increasing by looking at the slope of the tangent:

a) If the slope of the tangent is positive at $x = a$, then the function is increasing at $x = a$.

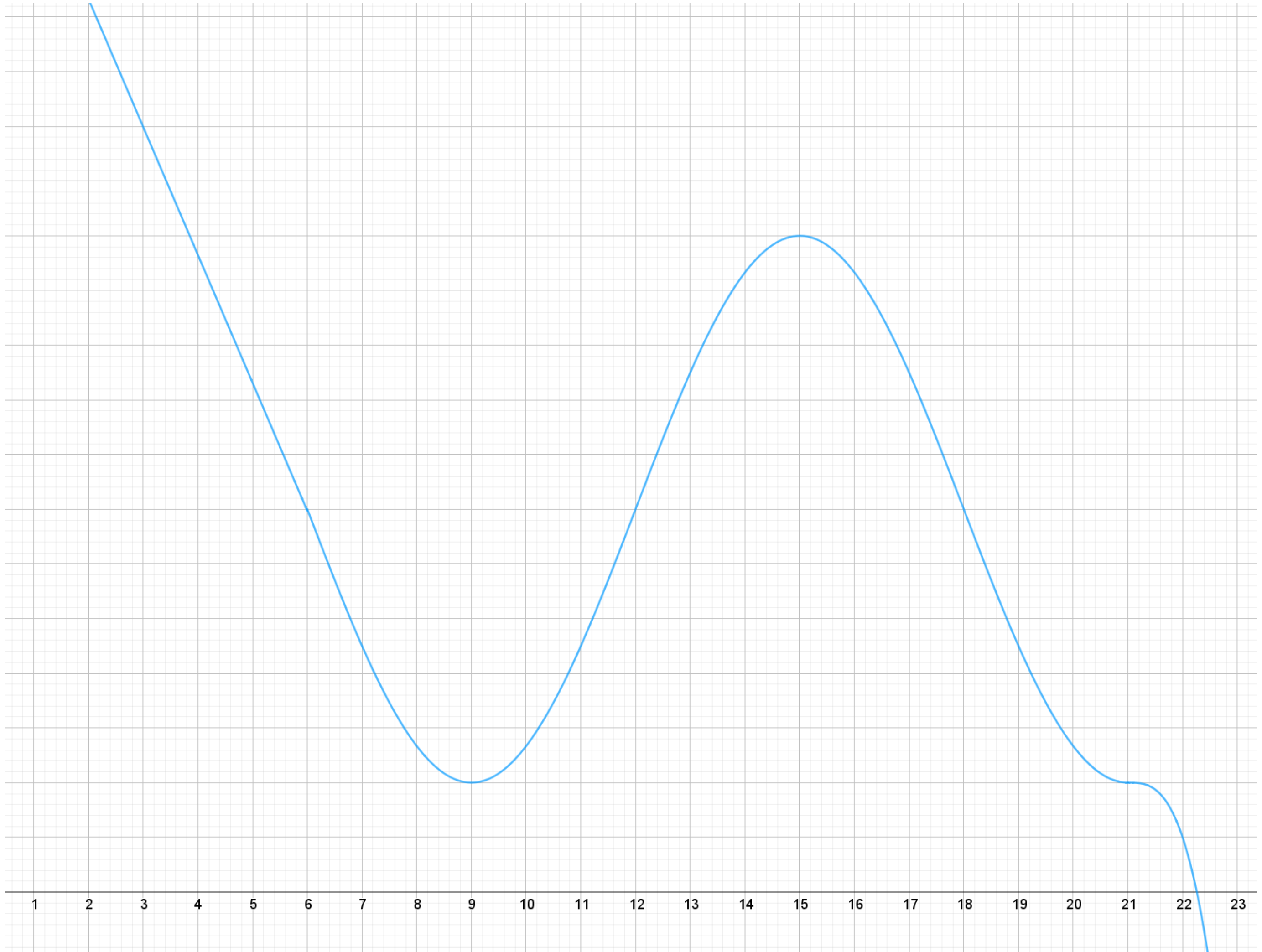
b) If the slope of the tangent is negative at $x = a$, then the function is decreasing at $x = a$.

c) If the slope of the tangent is = 0 or undefined at $x = a$, then it is unclear if the function is increasing or decreasing (or perhaps it is neither as the function might change from increasing to decreasing or vice versa).

Examples: Increasing and Decreasing

<u>Example 5:</u>

Consider the graph below.



a) Where is the function increasing?
b) Where is the function decreasing?
c) Indicate any points where the functions is neither increasing nor decreasing?

<u>Solution:</u>

a) Increasing: $(9,15)$
b) Decreasing: $(-\infty, 9) \cup (15, \infty)$
c) We note that the function is not increasing or decreasing at $x = 9$ and $x = 15$ as the graph changes from increasing to decreasing (or vice versa) at those points. We call these **<u>turning points</u>**.

<u>Important note:</u>
The slopes of the tangents are 0 (horizontal) at $x = 9{,}15$, and 21. However, the turning points only happen at $x = 9$ and $x = 15$. **<u>This means that just because the slope of the tangent is 0 it does not always mean we are on a turning point!</u>**

Examples: Increasing and Decreasing

Example 6:

Use the instantaneous rate of change to determine if the function is increasing or decreasing (if possible) at $x = -3$ and $x = 0$.

$$f(x) = x^3 + 3x^2$$

Solution:

$\underline{x = -3:}$

$a = -3,\ \ f(x) = x^3 + 3x^2,\ \ f(-3) = 0$

IROC $= \lim\limits_{x \to a} \frac{f(x) - f(a)}{x - a}$

$= \lim\limits_{x \to -3} \frac{f(x) - f(-3)}{x - (-3)}$

$= \lim\limits_{x \to -3} \frac{x^3 + 3x^2}{x + 3}$

$= \lim\limits_{x \to -3} \frac{x^2(x+3)}{x + 3}$

$= \lim\limits_{x \to -3} x^2$

$= 9$

$\therefore$ The function is increasing at $x = -3$

$\underline{x = 0:}$

$a = 0 \quad f(x) = x^3 + 3x^2 \quad f(0) = 0$

IROC $= \lim\limits_{x \to a} \frac{f(x) - f(a)}{x - a}$

$= \lim\limits_{x \to 0} \frac{f(x) - f(0)}{x - 0}$

$= \lim\limits_{x \to 0} \frac{x^3 + 3x^2}{x}$

$= \lim\limits_{x \to 0} \frac{x^2(x+3)}{x}$

$= \lim\limits_{x \to 0} x(x + 3)$

$= 0$

$\therefore$ We cannot use the IROC to determine if it is increasing or decreasing at $x = 0$.

Definition: Derivative

A derivative is a function that stores all of the instantaneous rates of change for a given function.

For example, say we had $f(x) = x^2$, and we wanted to calculate the instantaneous rate of change at

$$x = 2, 7, 13, 15, 21, -7, -30, \ldots$$

We could:

1) Find each IROC limit one at a time

2) Find one IROC limit that stores a function, and then substitute $x = 2, 7, 13, \ldots$ into the new function.

Choice (2) ends up saving quite a bit of work as we only need to find one limit, but the limit is a bit harder to solve.

We use "prime" notation to indicate the new name of the derivative function (denoted by $f'(x) = \cdots$ to indicate that $f'(a) = IROC(a)$ using f)

We also may use $\frac{d}{dx}f$ or $\frac{df}{dx}$ as the new name of the derivative function (which is read as df by dx or "the derivative of f with respect to x").

To find the derivative from first principles we simply leave a as it is rather than substituting in values of a.

Examples: Using the Definition

Example 1:

Find the derivative of $f(x) = x^2$ using the definition of a derivative, then use the derivative to find the instantaneous rate of change for $a = 2, a = 4, and\ a = -7$.

Solution:

$$
\begin{aligned}
f'(a) \quad &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\
&= \lim_{x \to a} \frac{x^2 - a^2}{x - a} \\
&= \lim_{x \to a} \frac{(x-a)(x+a)}{x - a} \\
&= \lim_{x \to a} x + a \\
&= 2a
\end{aligned}
$$

$\therefore f'(a) = 2a$ and so $f'(2) = 4, f'(4) = 8$ and $f'(-7) = -14$.

Examples: Using the Definition

Example 2:

Find the derivative of $f(x) = \sqrt{x}$ using the definition of a derivative, then use the derivative to find the instantaneous rate of change for $a = 4, a = 0, and\ a = -7$.

Solution:

$$
\begin{aligned}
f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\
&= \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a} \\
&= \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a} \left(\frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}} \right) \\
&= \lim_{x \to a} \frac{x - a}{(x - a)(\sqrt{x} + \sqrt{a})} \\
&= \lim_{x \to a} \frac{1}{\sqrt{x} + \sqrt{a}} \\
&= \frac{1}{2\sqrt{a}}
\end{aligned}
$$

$\therefore f'(a) = \frac{1}{2\sqrt{a}}$ and so $f'(2) = \frac{1}{4}, f'(0)$ is undefined (cannot divide by 0) and $f'(-7)$ is undefined (cannot square root a negative).

Examples: Using the Definition

Example 3:

Determine all x-values of $f(x) = x^3 + 2x^2$ that will have tangents parallel to the line y = 2.

Solution:

We first need to calculate a formula for the derivative:

$$
\begin{aligned}
f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\
&= \lim_{x \to a} \frac{x^3 + 2x^2 - (a^3 + 2a^2)}{x - a} \\
&= \lim_{x \to a} \frac{x^3 - a^3 + 2x^2 - 2a^2}{x - a} \\
&= \lim_{x \to a} \frac{(x-a)(x^2 + ax + a^2) + 2(x-a)(x+a)}{x - a} \\
&= \lim_{x \to a} \frac{(x-a)[(x^2 + ax + a^2) + 2(x+a)]}{x - a} \\
&= \lim_{x \to a} (x^2 + ax + a^2) + 2(x + a) \\
&= 3a^2 + 4a
\end{aligned}
$$

$\therefore f'(a) = 3a^2 + 4a$

Since we want to know where tangents are parallel to y = 2, we want to know when the slope of the tangent is 0. This means we look for $f'(a) = 0$ and solve for a:

$0 = 3a^2 + 4a$
$0 = a(3a + 4)$
$\therefore a = 0$ and $3a + 4 = 0$
Solving these gives: $a = 0, -\frac{4}{3}$.

Formula: Power Rule

Formula(s):	**When To Use it:**	
Given $f(x) = x^n$ $f'(x) = nx^{n-1}$ (That is to derive a power, we multiply the exponent in front of the term, and subtract one from the exponent). Note that this works for any constant exponent (as long as the exponent is a number). You may need to rewrite the expression using exponent laws before you start using the power rule.	When calculating a derivative where the base is a variable and the exponent is a number.	

Example 4:

Derive the following expressions: i) $f(x) = x^3$ ii) $g(x) = \sqrt{x}$ iii) $h(x) = \frac{1}{x}$

Solution:

i) $f(x) = x^3$

$\rightarrow f'(x) = 3x^{3-1}$

$\rightarrow f'(x) = 3x^2$

ii) $g(x) = \sqrt{x}$

$g(x) = x^{1/2}$

$\rightarrow g'(x) = \frac{1}{2}x^{1/2-1}$

$\rightarrow g'(x) = \frac{1}{2}x^{-1/2}$

iii) $h(x) = \frac{1}{x}$

$h(x) = x^{-1}$

$\rightarrow h'(x) = -1x^{-1-1}$

$\rightarrow h'(x) = -x^{-2}$

Formula: Constant Rule

Formula(s):	**When To Use it:**	**Proof of Formula(s)**
Given $f(x) = c$ $f'(x) = 0$ (That is to derive a constant, it is always zero). Note, this can get tricky if you are not careful! Watch out for e vs e^x and $\ln(\pi)$ vs $\ln(x)$.	When calculating a derivative of any constant number (ie, no x is present)	If you draw a constant function, it is a horizontal line. This always has a slope of 0 everywhere!

Example 5:

Derive the following expressions: i) $f(x) = 7$ ii) $g(x) = \pi^3$ iii) $h(x) = e^\pi$

Solution:

All of these are constant functions (remember, π and e are constants!). Thus they all derive to become 0.

Formula: Sum/Difference Rule

Formula(s):	When To Use it:	
If given $h(x) = f(x) + g(x)$ $h'(x) = f'(x) + g'(x)$ If given $h(x) = f(x) - g(x)$ $h'(x) = f'(x) - g'(x)$ (That is to derive a sum/difference, we derive each term separately and add/subtract the corresponding derivatives).	When calculating a derivative of the sum (or difference) of two different functions.	

Example 6:

Derive the following expression: $g(x) = x^2 + x + 7 - \frac{1}{x} + e$

Solution:

We derive each part separately:

x^2	derives to become	$2x^{2-1} = 2x$	(Power rule)
x	derives to become	$1x^{1-1} = 1$	(Power rule)
7	derives to become	0	(Constant rule)
$\frac{1}{x} = x^{-1}$	derives to become	$-1x^{-1-1} = -x^{-2}$	(Power rule)
e	derives to become	0	(Constant rule)

The sum/difference rule states that we can simply add/subtract all of these derivatives separately to get:

$$g'(x) = 2x + 1 + 0 - (-x^{-2}) + 0$$
$$= 2x + x^{-2} + 1$$

Formula: Coefficient Rule

Formula(s):	**When To Use it:**	
If given $h(x) = k\ f(x)$ $h'(x) = k\ f'(x)$ **(That is to derive an expression with a coefficient, simply ignore the coefficient, derive the other expression, then multiply the coefficient).**	**When calculating a derivative of an expression with a coefficient.**	

Example 7:

Derive the following expression: $g(x) = 4x^2 + 8x + 7 - \frac{10}{x} + e^4$

Solution:

We derive each part separately:

$4x^2$	derives to become	$4(2x^{2-1}) = 8x$	(Power rule with coefficient rule)
$8x$	derives to become	$8(1x^{1-1}) = 8$	(Power rule with coefficient rule)
7	derives to become	0	(Constant rule)
$\frac{10}{x} = 10x^{-1}$	derives to become	$10(-1x^{-1-1}) = -10x^{-2}$	(Power rule with coefficient rule)
e^4	derives to become	0	(Constant rule)

The sum/difference rule states that we can simply add/subtract all of these derivatives separately to get:

$$\begin{aligned} g'(x) &= 8x + 8 + 0 - (-10x^{-2}) + 0 \\ &= 8x + 10x^{-2} + 8 \end{aligned}$$

Formula: Sin, Cos, e and other exponentials Rule

Function	Derivative
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	sec^2x
$\csc x$	$-\csc x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-csc^2x$
e^x	e^x
a^x	$a^x \ln a$

Example 8:

Derive the following expression: $g(x) = \cos(x) - 3\sin(x) + 7e^x - 5(2^x) + \pi e + x^e$

Solution:

We derive each part separately:

$\cos(x)$	derives to become	-sin(x)	(Cos rule)
$3\sin(x)$	derives to become	$3(\cos(x))$(sin rule with coefficient rule)	
$7e^x$	derives to become	$7(e^x)$	(e^x rule with coefficient rule)
$5(2^x)$	derives to become	$5(2^x \ln(2))$	(a^x rule with coefficient rule)
πe	derives to become	0	(constant rule)
x^e	derives to become	ex^{e-1}	(Power rule **NOT** $\boldsymbol{a^x}$)

The sum/difference rule states that we can simply add/subtract all of these derivatives separately to get:

$$\begin{aligned} g'(x) &= -\sin(\mathrm{x}) - 3\cos(\mathrm{x}) + 7\mathrm{e^x} - 5(\ln(2))2^x + 0 + ex^{e-1} \\ &= -\sin(\mathrm{x}) - 3\cos(\mathrm{x}) + 7\mathrm{e^x} - 5(\ln(2))2^x + ex^{e-1} \end{aligned}$$

Formula: Product Rule

Formula(s):	**When To Use it:**	
If we have $h(x) = f(x)\, g(x)$ $h'(x) = f'(x)g(x) + f(x)g'(x)$ (That is we take the derivative of the first function times the original second function, then add the original first function times the derivative of the second function). Note: IT IS NOT TRUE THAT $h'(x) = f'(x)\, g'(x)$	When calculating a derivative that involves a product of two or more functions.	

Example 9:

Consider the table below:

x	-3	-1	0	2	5	7
$f(x)$	5	0	-2	8	3	0
$g(x)$	13	14	-4	1	-3	1
$f'(x)$	-10	9	5	6	3	2
$g'(x)$	7	-6	2	-4	-3	3

If $h(x) = f(x)g(x) + f(x)^2$ what is $h'(2)$?

Solution:

Here we have two products

For the first one we see that $\left(f(x)g(x)\right)' = f'(x)g(x) + f(x)g'(x)$ from the product rule.

For the second one we see that $(f(x)f(x))' = f'(x)f(x) + f(x)f'(x)$ from the product rule.

This means we add these two derivatives together (due to the sum rule) and get:

$$h'(x) = f'(x)g(x) + f(x)g'(x) + f'(x)f(x) + f(x)f'(x)$$

$$\begin{aligned} \therefore h'(2) &= f'(2)g(2) + f(2)g'(2) + f'(2)f(2) + f(2)f'(2) \\ &= (6)(1) + (8)(-4) + (6)(8) + (8)(6) \\ &= 6 - 32 + 48 + 48 \\ &= 70 \end{aligned}$$

Examples: Product Rule

Example 10:

Determine the derivatives of:

i) $f(x) = 2x^2 3^x$

ii) $f(x) = (\sin(x) + \cos(x))e^x$

iii) $f(x) = \sin(2x)$ (using identities)

Solution:

i) Here we need the product rule:

$A = 2x^2 \quad \mathrm{B} = 3^x$

$A' = 4x \quad B' = 3^x(\ln(3))$

$\therefore f'(x)$

$= A'B + AB'$

$= (4x)(3^x) + (2x^2)(3^x(\ln(3))$

$= 4x3^x + 2\ln(3)x^2 3^x$

ii) Using the product rule we see:

$A = \sin(x) + \cos(x) \quad B = \mathrm{e^x}$

$A' = \cos(x) - \sin(x) \quad B' = \mathrm{e^x}$

$\therefore f'(x)$

$= A'B + AB'$

$= (\cos(x) - \sin(x))(\mathrm{e^x}) + (\sin(x) + \cos(x))(e^x)$

$= \cos(x)\, e^x - \sin(x)\, e^x + \sin(x)\, e^x + \cos(x)\, e^x$

$= 2\cos(x)\, e^x$

iii) We recall that $\sin(2x) = 2\sin(x)\cos(x)$

$A = 2\sin(x) \qquad B = \cos(x)$

$A' = 2\cos(x) \qquad B' = -\sin(x)$

$\therefore f'(x)$

$= A'B + AB'$

$= 2\cos^2(x) - 2\sin^2(x)$

Formula: Quotient Rule

Formula(s):

If we have $h(x) = \frac{f(x)}{g(x)}$ Then we get $h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

(That is we take the derivative of the first function times the original second function, then subtract the original first function times the derivative of the second function, all divided by the square of the second function). Note: IT IS NOT TRUE THAT $h'(x) = \frac{f'(x)}{g'(x)}$

When To Use it:

When calculating a derivative that involves a quotient of two or more functions.

Example 11:

Consider the table below:

x	-3	-1	0	2	5	7
$f(x)$	5	0	-2	8	3	1
$g(x)$	13	14	-4	1	-3	0
$f'(x)$	-10	9	5	6	3	2
$g'(x)$	7	-6	2	-4	-3	3

i) If $h(x) = \frac{f(x)}{g(x)}$ what is $h'(2)$?

ii) Which derivative(s) for h would be undefined?

Solution:

i) Using the quotient rule we get:

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

$$\therefore h'(2) = \frac{f'(2)g(2) - f(2)g'(2)}{g(2)^2} = \frac{(6)(1) - (8)(-4)}{1^2} = 38$$

ii) To have an undefined derivative we would need a division by 0. In this case, we would want to find when $g(x) = 0$. This only happens at $x = 7$. This means $h'(7)$ would be undefined.

Examples: Quotient Rule

Example 12:

Determine the derivatives of:

i) $f(x) = \frac{2x^2-5}{e^x}$

ii) $f(x) = \tan x$ (using identities)

Solution:

i) Here we need the quotient rule:

$A = 2x^2 - 5$ $\quad B = e^x$

$A' = 4x$ $\quad B' = e^x$

$$\begin{aligned}
\therefore f'(x) &= \frac{A'B-AB'}{B^2} \\
&= \frac{(4x)(e^x)-(2x^2-5)(e^x)}{(e^x)^2} \\
&= \frac{(e^x)[(4x)-(2x^2-5)]}{(e^x)^2} \\
&= \frac{-2x^2+4x+5}{e^x}
\end{aligned}$$

ii) We should recall that $f(x) = \tan x = \frac{\sin x}{\cos x}$

$A = \sin(x)$ $\quad B = \cos(x)$

$A' = \cos(x)$ $\quad B' = -\sin(x)$

$$\begin{aligned}
\therefore f'(x) &= \frac{A'B-AB'}{B^2} \\
&= \frac{(\cos(x))(\cos(x))-(\sin(x))(-\sin(x))}{\cos(\mathrm{x})^2} \\
&= \frac{\sin^2 x+\cos^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= sec^2\, x
\end{aligned}$$

Formula: Chain Rule

Formula(s):	**When To Use it:**	
If we have $h(x) = f \; o \; g(x) = f\big(g(x)\big)$ Then we get $h'(x) = f'\big(g(x)\big)g'(x)$ (That is we take the derivative of the outer function leaving the inside untouched, then multiply the derivative of the inner function at the end). Note: it is often a good idea to use the box method to consider the chain rule. Decide what rule appears (power, e, sin, etc…) and place a box for the input. Then apply the chain rule remembering to include a $\square'$ at the end.	When calculating a derivative that involves a composition of two or more functions.	

Examples: Chain Rule

Example 1:

Consider the table below:

x	-1	0	2	3	5	7
$f(x)$	0	-2	8	5	7	1
$g(x)$	14	-4	-1	13	-3	0
$f'(x)$	9	5	6	-10	3	2
$g'(x)$	-6	2	-4	7	-3	3

i) If $h(x) = f\big(g(x)\big)$ what is $h'(2)$?

ii) If $k(x) = f\Big(f\big(f(x)\big)\Big)$, what is $k'(3)$?

Solution:

i) Using the chain rule we get:

$$h'(x) = f'\big(g(x)\big)g'(x)$$

$$\begin{aligned}\therefore h'(2) &= f'\big(g(2)\big)g'(2)\\ &= f'(-1)g'(2)\\ &= (9)(-4)\\ &= -36\end{aligned}$$

ii) Using the chain rule twice we get:

$$\begin{aligned}k'(x) &= f'(f(f(x)))\big[f\big(f(x)\big)\big]'\\ &= f'(f(f(x)))f'\big(f(x)\big)f'(x)\end{aligned}$$

$$\begin{aligned}\therefore k'(3) &= f'(f(f(3)))f'\big(f(3)\big)f'(3)\\ &= f'(f(5))f'(5)f'(3)\\ &= f'(7)f'(5)f'(3)\\ &= (2)(3)(-10)\\ &= -60\end{aligned}$$

Examples: Chain Rule

Example 2:

Find the derivative of $f(x) = (x + e^x)^{10}$

Solution:

Here we see that we are deriving $\square^{10}$ which means we use the chain rule:

$$
\begin{aligned}
f(x) &= (\square)^{10} \\
f'(x) &= [\square^{10}]' \\
&= 10\square^9[\square]' \\
&= 10(x + e^x)^9[x + e^x]' \\
&= 10(x + e^x)^9(1 + e^x)
\end{aligned}
$$

Examples: Chain Rule

Example 3:

Find the derivative of $f(x) = e^{e^{x^2}}$

Solution:

Here we see that we are deriving $e^{\square}$ which means we use the chain rule:

$$
\begin{aligned}
f(x) &= e^{\square} \\
f'(x) &= [e^{\square}]' \\
&= e^{\square}[\square]' \\
&= e^{e^{x^2}}\left[e^{x^2}\right]' \\
&= e^{e^{x^2}}[e^{\square}]' \\
&= e^{e^{x^2}}(e^{\square})[\square]' \\
&= e^{e^{x^2}}\left(e^{x^2}\right)[x^2]' \\
&= e^{e^{x^2}}\left(e^{x^2}\right)(2x) \\
&= 2xe^{e^{x^2}+x^2}
\end{aligned}
$$

Examples: Chain Rule

Example 4:

Find the derivative of $f(x) = \cos^{-4}(\sqrt{x})$. Note that this is not arc cos!!

Solution:

We rewrite the function first to get:

$$
\begin{aligned}
f(x) &= \left(\cos\left(x^{\frac{1}{2}}\right)\right)^{-4} \\
&= \square^{-4} \\
f'(x) &= [\square^{-4}]' \\
&= -4[\square]^{-5}[\square]' \\
&= -4\left(\cos\left(x^{\frac{1}{2}}\right)\right)^{-5}\left[\cos\left(x^{\frac{1}{2}}\right)\right]' \\
&= -4\cos^{-5}\left(x^{\frac{1}{2}}\right)[\cos\square]' \\
&= -4\cos^{-5}\left(x^{\frac{1}{2}}\right)(-\sin(\square))[\square]'
\end{aligned}
$$

$$
\begin{aligned}
&= 4\cos^{-5}\left(x^{\frac{1}{2}}\right)\left(\sin\left(x^{\frac{1}{2}}\right)\right)\left[x^{\frac{1}{2}}\right]' \\
&= 4\cos^{-5}\left(x^{\frac{1}{2}}\right)\left(\sin\left(x^{\frac{1}{2}}\right)\right)\left(\frac{1}{2}x^{-\frac{1}{2}}\right) \\
&= 2x^{-\frac{1}{2}}\cos^{-5}\left(x^{\frac{1}{2}}\right)\left(\sin\left(x^{\frac{1}{2}}\right)\right)
\end{aligned}
$$

Strategy: Combining Multiple Derivative Rules

Strategy):	When To Use it:	Proof of Formula(s)
1) Identify the last operation you would need to perform, then this is the first differentiation rule you need to apply. 2) It helps to use shapes (for chain rules) and/or letters (for combinations of functions) to keep together the pieces you will need to derive next. 3) Repeat the above process one at a time until you have identified all derivatives.	When calculating a derivative that involves multiple derivative rules.	This strategy is here to help visualize the pieces that we need to handle. The rest is deriving the functions using other derivative rules we have proven.

Examples: Combining Multiple Rules

Example 5:

Derive: $f(x) = \sin(2x^2)\tan(x) + 2e^{4x}$

Solution:

The last operation we would need to perform (say we were inputting one operation at a time in the calculator) is the addition.

$f(x) \quad = A + B \qquad (A = \sin(2x^2)\tan(x),\ \ B = 2e^{4x}\ )$

$\therefore f'(x) \quad = A' + B' \qquad$ (Sum rule)

$= (CD)' + B' \qquad (C = \sin(2x^2),\quad D = \tan(x)\ )$

$= (C'D + CD') + B' \qquad$ (Product rule)

Now if we can figure our the derivatives of B, C, and D we are done!

$B = 2e^{4x}$

$\rightarrow B = 2e^{\square}$

$\rightarrow B' = 2e^{\square}(\square') \qquad$ (Coefficient rule says ignore coefficient of 2, and e rule with chain)

$\rightarrow B' = 2e^{4x}([4x]')$

$\rightarrow B' = 2e^{4x}(4) \qquad$ (Coefficient rule says ignore coefficient of 4, and power rule)

$\rightarrow B' = 8e^{4x}$

Examples: Combining Multiple Rules

Example 5: (continued)

Derive: $f(x) = \sin(2x^2)\tan(x) + 2e^{4x}$

Solution:

$C = \sin(2x^2)$

$\rightarrow C = \sin(\square)$

$\rightarrow C' = \cos(\square)\,\square'$ (Sin rule with chain)

$\rightarrow C' = \cos(2x^2)\,[2x^2]'$

$\rightarrow C' = \cos(2x^2)\,(4x)$ (Coefficient rule says ignore coefficient of 2, and power rule)

$\rightarrow C' = 4x\cos(2x^2)$

$D = \tan(x)$

$\rightarrow D' = \sec^2(x)$ (Arctan rule)

Putting this altogether gets us:

$f'(x) \quad = (C'D + CD') + B'$

$\quad = (4x\, cos(2x^2)\tan(x) + \sin(2x^2)\,(\sec^2(x))) + 8e^{4x}$

Examples: Combining Multiple Rules

Example 6:

Derive: $f(x) = (x^4 - cos(x))^{10}(\sin(2^x)\,)^{-4}$

Solution:

The last operation we would need to perform (say we were inputting one operation at a time in the calculator) is the multiplication.

$f(x) \quad = A\,B$ $\qquad (A = (x^4 - cos(x))^{10},\ \ B = (\sin(2^x))^{-4}\ )$

$\therefore f'(x) \quad = A'B + AB'$ $\qquad$ (Product rule)

$= (C^{10})'B + A(D^{-4})'$ $\qquad (C = x^4 - \cos(x)\,, D = \sin(2^x)\ \ )$

$= (10C^9C')B + A(-4D^{-5}\,D')$ $\qquad$ (Power rules with chain)

$= (10C^9(E-F)')B + A(-4D^{-5}\,(\sin(G))')$ $\qquad (E = x^4, F = \cos(x)\,, G = 2^{\mathrm{x}})$

$= (10C^9(E'-F'))B + A(-4D^{-5}\,(\cos(G))G')$ $\qquad$ (Difference rule and sin rule with chain)

$= \left(10C^9(4x^3 + \sin(x))\right)B + A\left(-4D^{-5}\,(\cos(G))(2^x \ln 2)\right)$ $\qquad$ (Power Cos and exponential rule)

$= \left(10(x^4 - \cos(x))^9(4x^3 + \sin(x))\right)(\sin(2^x))^{-4} + (x^4 - cos(x))^{10}(-4(\sin(2^x))^{-5}\cos(2^x)\,2^x\ln(2))$

Examples: Combining Multiple Rules

Example 7:

Derive: $f(x) = \frac{\sin\left(x^2 \cos(x)\right)}{2^{3^x}}$

Solution:

$f(x) \quad = \frac{A}{B}$ $\qquad$ $\left(A = \sin(x^2 \cos(x)),\quad B = 2^{3^x}\right)$

$\therefore f'(x) \quad = \frac{A'B - AB'}{B^2}$ $\qquad$ (Quotient rule)

$= \frac{(\sin(C))'B - A(2^D)'}{B^2}$ $\qquad$ $\left(C = x^2 \cos(x), D = 3^x\right)$

$= \frac{\left(\cos(C)\, C'\right)B - A\left(2^D \ln(2)\, D'\right)}{B^2}$ $\qquad$ (Sin and exponential rule with chain)

$= \frac{\left(\cos(C)\, (EF)'\right)B - A\left(2^D \ln(2)\, D'\right)}{B^2}$ $\qquad$ $\left(E = x^2, F = \cos(x)\right)$

$= \frac{\left(\cos(C)\, \left(E'F + EF'\right)\right)B - A\left(2^D \ln(2)\, D'\right)}{B^2}$ $\qquad$ (Product rule)

$= \frac{\left(\cos\left(x^2 \cos(x)\right) \left((2x)(\cos(x)) - x^2(\sin(x))\right)\right)2^{3^x} - \sin\left(x^2 \cos(x)\right)\left(2^{3^x} \ln(2)\, 3^x \ln(3)\right)}{\left(2^{3^x}\right)^2}$ $\qquad$ (Finalizing all remaining derivatives)

Definition: Horizontal and Undefined Tangents

A horizontal tangent line occurs when a line has a derivative that is 0.

A undefined tangent has one of three types:

1) The function is discontinuous.
2) The left and right derivatives do not meet at the same value (i.e. an abrupt change in slope).
3) The left and/or right derivatives approaches an infinity

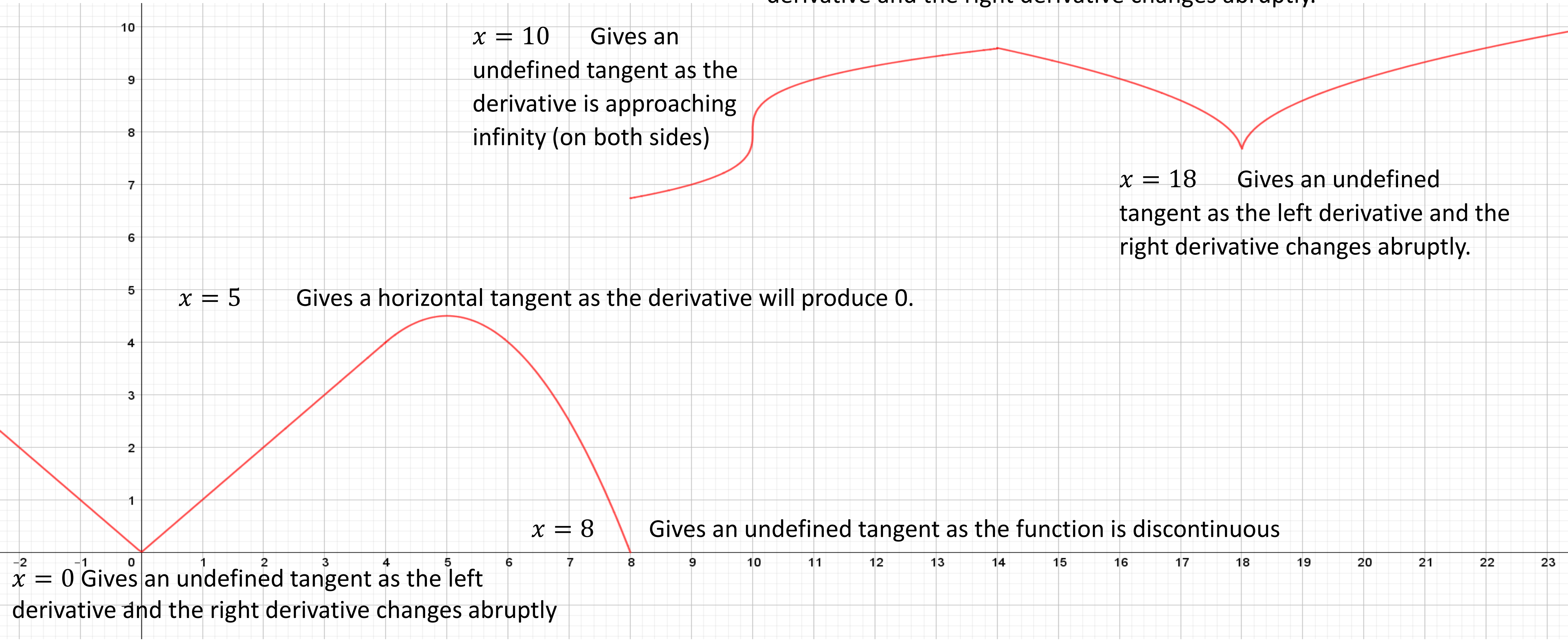
Examples: Horizontal and Undefined Tangents

Example 8:

In the graph below, determine the x-values where we have:

a) Horizontal Tangents

b) Undefined Tangents that are present.



Examples: Horizontal and Undefined Tangents

Example 7:

Find the derivative of the following function and identify any horizontal and undefined tangents: $f(x) = 3\sqrt[3]{x} - x$

Solution

We first note that this function is continuous everywhere (we can cube root positives, negatives, and 0). So there are no undefined tangents due to discontinuities. When we derive the function, we get:

$$\begin{aligned} f(x) &= 3\sqrt[3]{x} - x \\ &= 3x^{\frac{1}{3}} - x \\ \therefore f'(x) &= \tfrac{1}{3}(3)x^{-2/3} - 1 \\ &= \frac{1}{\sqrt[3]{x^2}} - 1 \end{aligned}$$

Here we see that the derivative is undefined when $x = 0$ as this will produce a 0 in the denominator.

We can also find any horizontal tangents by solving when the derivative is 0:

$$0 = \frac{1}{\sqrt[3]{x^2}} - 1$$

$$1 = \frac{1}{\sqrt[3]{x^2}}$$

$$\sqrt[3]{x^2} = 1$$

$$x^2 = 1^3$$

$$x = \pm\sqrt{1}$$

$$x = \pm 1$$

Thus the function has horizontal tangents at $x = -1$ and at $x = 1$.

Formula: Inverse Rule

Formula(s):

If we have an inverse function $f^{-1}(x)$. Then the derivative would give:

$$[f^{-1}(x)]' = \frac{1}{f'\left(f^{-1}(x)\right)}$$

That is the derivative of an inverse function is 1 divided by the derivative of the original function with an input of the inverse function.

When To Use it:

When calculating a derivative knowing what the inverse function is.

Example 1:

Consider the table below:

x	-1	0	2	3	5	7
$h(x)$	0	-2	8	5	7	3
$g(x)$	14	-4	-1	13	-3	0
$h'(x)$	9	5	6	-10	3	2
$g'(x)$	-6	2	-4	7	-3	3

Find $[h^{-1}(x)]'$ evaluated when $x = 3$. Assume that h^{-1} exists using the table above.

Solution:

Using the inverse rule we get:

$$\begin{aligned}[h^{-1}(x)]' &= \frac{1}{h'\left(h^{-1}(x)\right)} \\ &= \frac{1}{h'\left(h^{-1}(3)\right)}\end{aligned}$$

Since we are looking for $h^{-1}(3) = x$, we are actually looking for what x value gives $3 = h(x)$.

Looking at the table, we should see that $x = 7$, which means $h^{-1}(3) = x = 7$.

$$\begin{aligned}\frac{1}{h'\left(h^{-1}(3)\right)} &= \frac{1}{h'(7)} \\ &= \frac{1}{2}\end{aligned}$$

Examples: Inverse Formula

Example 2:

Find the derivative of $f^{-1}(x) = \ln x$ using the inverse rule.

Solution:

Since we know that $f^{-1}(x)$ has an inverse of $f(x) = e^x$, we can use the inverse formula:

Note: $f'(x) = [e^x]' = e^x$

$$
\begin{aligned}
[f^{-1}(x)]' &= \frac{1}{f'\left(f^{-1}(x)\right)} \\
&= \frac{1}{f'(\ln(x))} \\
&= \frac{1}{e^{\ln(x)}} \\
&= \frac{1}{x}
\end{aligned}
$$

Formula: ln, log, and arc functions

Function	Derivative
$[\ln(x)]'$	$\frac{1}{x}$
$[log_a(x)]'$	$\frac{1}{xln(a)}$
$[\arcsin(x)]'$	$\frac{1}{\sqrt{1-x^2}}$
$[\arccos(x)\ ]'$	$\frac{-1}{\sqrt{1-x^2}}$
$[\arctan\ (x)\ ]'$	$\frac{1}{1+x^2}$
$[\mathrm{arccsc}\ (x)]'$	$\frac{-1}{x\sqrt{x^2-1}}$
$[\mathrm{arcsec}\ (x)]'$	$\frac{1}{x\sqrt{x^2-1}}$
$[\mathrm{arccot}\ (x)\ ]'$	$\frac{-1}{1+x^2}$

Examples: Inverse Function Derivatives

Example 3:

Find the derivative of $f(x) = \frac{\arctan(x)}{2\log_2(x)} - \arcsin(x)$ (do not simplify)

Solution:

Here we see that we have a quotient rule and a subtraction rule. Since the subtraction would be the last one to plug into a calculator, it is our first derivative rule:

$$f(x) \quad = A - B \qquad \left(A = \frac{\arctan(x)}{2\log_2(x)}, B = \arcsin(x)\right)$$

$$f'(x) \quad = A' - B'$$

$$= \left(\frac{C}{D}\right)' - B' \qquad (C = \arctan(x), D = 2\log_2(x))$$

$$= \frac{C'D - CD'}{D^2} - B'$$

$$= \frac{\frac{1}{1+x^2}(2\log_2 x) - \arctan(x)\left(\frac{2}{x\ln(2)}\right)}{4(\log_2 x)^2} - \frac{1}{\sqrt{1-x^2}}$$

Examples: Inverse Function Derivatives

Example 4:

Find the derivative of $f(x) = \log_2(\arccos\left(3^{arcsin(2x)}\right)$ (do not simplify)

Solution:

We see that we have many chain rules. Using the box method we get:

$$f(x) = \log_2 \square \qquad \left(\square = \arccos\left(3^{arcsin(2x)}\right)\right)$$

$$f'(x) = \frac{1}{\square \ln(2)} \square' \qquad \text{(log rule)}$$

$$= \frac{1}{\arccos\left(3^{arcsin(2x)}\right)\ln(2)} \left[\arccos\left(3^{arcsin(2x)}\right)\right]'$$

$$= \frac{1}{\arccos\left(3^{arcsin(2x)}\right)\ln(2)} [\arccos(\square)]' \qquad \left(\square = 3^{arcsin(2x)}\right)$$

$$= \frac{1}{\arccos\left(3^{arcsin(2x)}\right)\ln(2)} \left(-\frac{1}{\sqrt{1-\square^2}}\right) [\square]' \qquad \text{(arc cos rule)}$$

$$= \frac{1}{\arccos\left(3^{arcsin(2x)}\right)\ln(2)} \left(-\frac{1}{\sqrt{1-3^{arcsin(2x)}{}^2}}\right) \left[3^{arcsin(2x)}\right]' \qquad \text{(arc cos rule)}$$

$$= \frac{1}{\arccos\left(3^{arcsin(2x)}\right)\ln(2)} \left(-\frac{1}{\sqrt{1-3^{arcsin(2x)}{}^2}}\right) [3^{\square}]' \qquad (\square = \arcsin(2x))$$

$$= \frac{1}{\arccos\left(3^{arcsin(2x)}\right)\ln(2)} \left(-\frac{1}{\sqrt{1-3^{arcsin(2x)}{}^2}}\right) 3^{\square} \ln(3)\, \square' \qquad (a^x \text{ rule})$$

Examples: Inverse Function Derivatives

Example 4 (continued):

Find the derivative of $f(x) = \log_2(\arccos(3^{arcsin(2x)})$ (do not simplify)

Solution:

$$= \frac{1}{\arccos(3^{arcsin(2x)})\ln(2)}\left(-\frac{1}{\sqrt{1-3^{arcsin(2x)^2}}}\right)3^{\text{arcsin}(2x)}\ln(3)\,[\text{arcsin}(2x)]'$$

$$= \frac{1}{\arccos(3^{arcsin(2x)})\ln(2)}\left(-\frac{1}{\sqrt{1-3^{arcsin(2x)^2}}}\right)3^{\text{arcsin}(2x)}\ln(3)\,[\text{arcsin}(\square)]'$$ ($\square$ = 2x)

$$= \frac{1}{\arccos(3^{arcsin(2x)})\ln(2)}\left(-\frac{1}{\sqrt{1-3^{arcsin(2x)^2}}}\right)3^{\text{arcsin}(2x)}\ln(3)\left(\frac{1}{\sqrt{1-\square^2}}\right)\square'$$ (arcsin rule)

$$= \frac{1}{\arccos(3^{arcsin(2x)})\ln(2)}\left(-\frac{1}{\sqrt{1-3^{arcsin(2x)^2}}}\right)3^{\text{arcsin}(2x)}\ln(3)\left(\frac{1}{\sqrt{1-(2x)^2}}\right)2$$ (power rule with coefficient)

Examples: Inverse Function Derivatives

Example 5:

Find the derivative of $f(x) = \ln(x^2 + sin(x))\, e^{\arccos(\sqrt{x}+4)}$

Solution:

We first note that the product is the last operation we would punch into a calculator, thus it is our first derivative rule:

$$f(x) = AB \qquad (A = \ln(x^2 + sin(x))\,, B = e^{\arccos(\sqrt{x}+4)})$$

$$f'(x) = A'B + AB'$$

$$= [\ln(C)]'B + A[e^D]' \qquad (C = x^2 + sin(x)\,, D = \arccos(\sqrt{x} + 4))$$

$$= \frac{1}{C}[C]'B + Ae^D[D]'$$

$$= \frac{1}{C}[2x + \cos(x)]B + Ae^D(\arccos(E))' \qquad (E = \sqrt{x} + 4)$$

$$= \frac{1}{C}[2x + \cos(x)]B + Ae^D\left(-\frac{1}{\sqrt{1-E^2}}\right)[E]'$$

$$= \frac{1}{C}[2x + \cos(x)]B + Ae^D\left(-\frac{1}{\sqrt{1-E^2}}\right)\left(\frac{1}{2}x^{-1/2}\right)$$

$$= \frac{1}{x^2+sin(x)}[2x + \cos(x)]e^{\arccos(\sqrt{x}+4)} + \ln(x^2 + sin(x))\, e^{\arccos(\sqrt{x}+4)}\left(-\frac{1}{\sqrt{1-(\sqrt{x}+4)^2}}\right)\left(\frac{1}{2}x^{-1/2}\right)$$

Formula: Deriving Absolute Value Functions

Formula(s):

The derivative of $|x|$ is $\frac{x}{|x|}$. This this means with the chain rule we get:

$$|\square|' = \frac{\square}{|\square|}\square'$$

When To Use it:

When deriving functions involving absolute value functions.

Proof of Formula(s)

This idea works as we have

$$\sqrt{\square^2} = |\square|$$

If we derive using the chain rule twice we see that we get:

$$\begin{aligned} |\square|' &= \sqrt{\square^2}' \\ &= \frac{1}{2\sqrt{\square^2}}(2\square)\square' \\ &= \frac{\square}{|\square|}\square' \end{aligned}$$

(The last step simplifies since $\sqrt{\square^2} = |\square|$)

Examples: Absolute Value Derivatives

Example 6:

Find the derivative of the following using the definition that $\sqrt{\square^2} = |\square|,\ f(x) = |x|$

Solution:

$$
\begin{aligned}
f(x) &= |x| \\
&= \sqrt{x^2} \\
&= \square^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
f'(x) &= \frac{1}{2}\square^{-1/2}\,\square' \\
&= \frac{1}{2\sqrt{x^2}}(2x) \\
&= \frac{x}{\sqrt{x^2}} \\
&= \frac{x}{|x|} \qquad \text{(as } |x| = \sqrt{x^2}\text{)}
\end{aligned}
$$

We note that $x = 0$ is not defined in the derivative.

Examples: Absolute Value Derivatives

Example 7:

Find the derivative of $f(x) = |3 - 5\sin(x)| + e^{\cos x}$

Solution:

We see that we the sum is the last operation we would punch into a calculator; thus it is our first operation to derive:

$$f(x) = A + B \qquad (A = |3 - 5\sin(x)|, B = e^{\cos x})$$

$$f'(x) = A' + B'$$

$$= [|C|]' + [e^D]' \qquad (C = 3 - 5\sin(x), D = \cos(x))$$

$$= \frac{C}{|C|}C' + e^D D'$$

$$= \frac{(3-5\sin(x))(-5\cos(x))}{|3-5\sin(x)|} - e^{\cos(x)}(\sin(x))$$

Formula: Deriving Piecewise Functions

Formula(s):	When To Use it:	Proof of Formula(s)
To derive a piecewise function, we must: 1) Derive each continuous piece separately 2) For the points which the function changes, we must check if it is differentiable using the definition of a derivative. $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$ Note that we will usually need to check two limits (one on either side of a) as the function changes at $x = a$	When calculating a derivative of a piecewise function.	We require the definition at points where the function changes to check for abrupt changes in the derivative which would make it undefined.

Examples: Piecewise Function Derivatives

Example 8:

Determine the derivative of $f(x) = \begin{cases} x & x \leq 0 \\ x+2 & x > 0 \end{cases}$

Solution:

Here we see that if we derive each piece separately, we get: $f'(x) = \begin{cases} 1 & x < 0 \\ 1 & x > 0 \end{cases}$

However, we must check $x = 0$ separately: $f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$

Since we have two different functions, we need to calculate each side separately:

$$\begin{aligned} \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \to 0^+} \frac{x + 2 - 0}{x - 0} \\ &= \lim_{x \to 0^+} \frac{x+2}{x} \\ &= \lim_{x \to 0^+} \frac{0^+ + 2}{0^+} \\ &= \lim_{x \to 0^+} \frac{2^+}{0^+} \\ &= \infty \end{aligned}$$

Thus the derivative does not exist at $x = 0$ as the right hand derivative does not exist (tends to an infinity). This means our derivative for the function is:

$f'(x) = \begin{cases} 1 & x < 0 \\ 1 & x > 0 \end{cases}$ (note that $x = 0$ is undefined)

Examples: Piecewise Function Derivatives

Example 9:

Determine the derivative of $f(x) = \begin{cases} x^2 + 1 & x \leq 1 \\ 2x & x > 1 \end{cases}$

Solution:

Here we see that if we derive each piece separately, we get: $f'(x) = \begin{cases} 2x & x < 1 \\ 2 & x > 1 \end{cases}$

However, we must check $x = 1$ separately: $f'(1) \quad = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$

Since we have two different functions, we need to calculate each side separately:

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{2x - 2}{x - 1} = \lim_{x \to 1^+} \frac{2(x - 1)}{x - 1} = 2$$

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{x^2 + 1 - 2}{x - 1} = \lim_{x \to 1^-} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1^-} \frac{(x - 1)(x + 1)}{x - 1} = \lim_{x \to 1^-} x + 1 = 2$$

Thus the derivative does exist at $x = 1$ and is 2. This gives us the derivative: $f'(x) = \begin{cases} 2x & x < 1 \\ 2 & x \geq 1 \end{cases}$

Examples: Piecewise Function Derivatives

Example 10:

Find the value of a that would make the function continuous at $x = 0$. Would this value allow for the function to be differentiable at $x = 0$?

$$f(x) = \begin{cases} x^2 sin\left(\frac{1}{x}\right) & x \neq 0 \\ a & x = 0 \end{cases}$$

Solution:

To allow for the function to be continuous at $x = 0$, we need $\lim_{x \to 0} f(x) = f(0)$

We find the limit first: $\lim_{x \to 0} f(x) \quad = \lim_{x \to 0} \mathrm{x}^2 \sin\left(\frac{1}{x}\right)$

Here we can use the squeeze theorem to note that : $\lim_{x \to 0} -x^2 \leq \lim_{x \to 0} \mathrm{x}^2 \sin\left(\frac{1}{x}\right) \leq \lim_{x \to 0} x^2$

Thus by the squeeze theorem, this limit must go to 0. This means we must define $a = 0$ for the function to be continuous.

We now check if the function is differentiable at $x = 0$:

$$\begin{aligned} f'(0) &= \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x \to 0} \frac{x^2 \sin\left(\frac{1}{x}\right) - 0}{x} \\ &= \lim_{x \to 0} \frac{x^2 \sin\left(\frac{1}{x}\right)}{x} \\ &= \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) \end{aligned}$$

Here we can use the squeeze theorem to note that :

$$\lim_{x \to 0} -x \leq \lim_{x \to 0} \mathrm{x} \sin\left(\frac{1}{x}\right) \leq \lim_{x \to 0} x$$

Thus by the squeeze theorem, this limit must go to 0. This means if we define $a = 0$ we would also get that the function is differentiable at $x = 0$ with a derivative $f'(0) = 0$

Strategy: Implicit Differentiation

Strategy:	When To Use it:	Proof of Formula(s)
If we have a relation where y appears on both sides and we wish to differentiate, we can: 1) Derive both sides using standard differentiation rules. <u>Be careful to include y'</u>, but you may use the fact that <u>$x' = 1$</u>. 2) Gather all y' terms to one side of the equation, and all other terms (terms only containing x and y) onto the other side. 3) Factor the side with y', then you can isolate y' by dividing the expression over to the other side.	When we cannot isolate for y (for example, when the relation is not a function), and we wish to find y'.	This is simply using our rules but thinking of differentiation as an "operation" we can apply on both sides of the equation. The reason why y' stays yet $x' = 1$ is because we are deriving with respect to x which means $\frac{dy}{dx} = y'$ must stay but $\frac{dx}{dx} = 1$.

Examples: Implicit Differentiation

Example 1:

Consider the relation: $x^2 + y^2 = 25$

a) Determine y' for the equation of a circle with radius 5 using implicit differentiation.

b) Use the derivative to determine the point(s) of vertical and horizontal tangents.

Solution:

We derive both sides at the same time to get:

$$[x^2]' + [y^2]' = [25]'$$
$$2x + 2yy' = 0$$

We then isolate for y':

$$2yy' = -2x$$
$$y' = -\frac{2x}{2y}$$
$$y' = -\frac{x}{y}$$

Horizontal tangents occur when $y' = 0$. This can only happen when the numerator is zero, thus we get horizontal tangents when $x = 0$. Subbing this into the equation will allow us to find our points:

$x^2 + y^2 = 25 \quad \rightarrow y^2 = 25 \quad \rightarrow y = \pm 5$ Thus we have two points where there are horizontal tangents (0, 5) and $(0, -5)$

Vertical tangents occur when the derivative is undefined. This happens when the denominator is zero, thus we get vertical tangents when $y = 0$. Subbing this into the original equation will allow us to find our points.

$x^2 + y^2 = 25 \quad \rightarrow x^2 = 25 \quad \rightarrow x = \pm 5$ Thus we have two points where there are vertical tangents (5, 0) and $(-5,0)$

Examples: Implicit Differentiation

Example 2:

Find y' from the following relation using implicit differentiation: $\sin(x)\cos(y) - e^{xy} = y$

Solution:

We derive both sides at the same time to get: $[\sin(x)\cos(y)]' - [e^{xy}]' = [y]'$

We cannot forget to use the product rule and chain rule when they apply!

$A'B + AB' - e^{\square}\square' = y'$

$\cos(x)\cos(y) + \sin(x)\,(-\sin(y))(y') - e^{xy}[xy]' = y'$

$\cos(x)\cos(y) - \sin(x)\sin(y)(y') - e^{xy}((1)y + xy') = y'$

$\cos(x)\cos(y) - \sin(x)\sin(y)\,(y') - e^{xy}y - e^{xy}xy' = y'$

Next we isolate for y':

$\cos(x)\cos(y) - e^{xy}y = y' + \sin(x)\sin(y)\,(y') + e^{xy}xy'$

$\cos(x)\cos(y) - e^{xy}y = [1 + \sin(x)\sin(y) + e^{xy}x]y'$

$\frac{\cos(x)\cos(y) - e^{xy}y}{1 + \sin(x)\sin(y) + e^{xy}x} = y'$

Examples: Implicit Differentiation

Example 3:

Find the equation of the tangent line at the point $(1,2)$ on the function: $x^2 + xy - y^2 = -1$

Solution:

We derive both sides at the same time to get: $[x^2]' + [xy]' - [y^2]' = [-1]'$

$2x + y + xy' - 2yy' = 0$

Our goal is to find y' when $x = 1$ and $y = 2$. We could solve for y' first, but it is usually easier to substitute in the point first:

$2(1) + 2 + (1)y' - 2(2)y' = 0$

$4 - 3y' = 0$

$y' = \frac{4}{3}$

This tells us the slope of the tangent line, we then sub into $y = mx + b$ to find the equation:

$y = \frac{4}{3}x + b$

$2 = \frac{4}{3}(1) + b$

$2 - \frac{4}{3} = b$

$\frac{2}{3} = b$

$\therefore$ The equation of the line on the curve at the point $(1,2)$ is $y = \frac{4}{3}x + \frac{2}{3}$

Strategy: Logarithmic Differentiation

Strategy:	When To Use it:	Proof of Formula(s)
If we have functions of the form $f(x)^{g(x)}$ or if we have a function with many products/quotients, we can do the following to derive: 1) Ln both sides of the equation 2) Simplify the result using log laws. 3) Use implicit differentiation to derive and find y' 4) If we were given an expression for y, don't forget to sub in y so the derivative is only in terms of x.	If we have functions of the form $f(x)^{g(x)}$ or if we have a function with many products/quotients.	Applying ln to both sides gives us an opportunity to change exponentiation into products and products/quotients into sums/differences. This can simplify the expression to allow us to differentiate.

Examples: Logarithmic Differentiation

Example 4:

Use logarithmic differentiation to find y' for the following function: $y = x^x$

Solution:

We first ln both sides and simplify the result:

$\ln(y) = \ln(x^x)$

$\ln(\mathrm{y}) = \mathrm{x}\ln(\mathrm{x})$

We can now use implicit differentiation to find the derivative y':

$[\ln(y)]' = [x\ln(x)]'$

$\frac{1}{y}y' = (1)\ln(x) + x\left(\frac{1}{x}\right)$

$\frac{1}{y}y' = \ln x + 1$

$y' = y(\ln(x) + 1)$

$y' = (x^x)(\ln(x) + 1)$

Examples: Logarithmic Differentiation

Example 5:

Use logarithmic differentiation to derive the following function: $f(x) = \frac{e^x(x+\sin(x))^{10}(\ln(x)+\pi)^{13}}{\arcsin(x)\tan(x)2^x}$

Solution:

At first glance, this would require 2 triple product rules and a quotient rule with many chain rules. However, if we use logarithmic differentiation, this becomes a lot easier to derive. We first ln both sides and simplify:

$$
\begin{aligned}
\ln(f(x)) &= \ln\left(\frac{e^x(x+\sin(x))^{10}(\ln(x)+\pi)^{13}}{\arcsin(x)\tan(x)2^x}\right) \\
&= \ln(e^x) + \ln(x+\sin(x))^{10} + \ln(\ln(x)+\pi)^{13} - \ln(arcsin(x)) - \ln(tan(x)) - \ln(2^x) \\
&= \text{x} + 10\ln(x+\sin(x)) + 13\ln(\ln(x)+\pi) - \ln(arcsin(x)) - \ln(tan(x)) - x\ln(2)
\end{aligned}
$$

Next we derive each side using implicit differentiation. It helps to recall that $\ln(\square)$ becomes $\frac{1}{\square}\square'$ when deriving.

$$
\begin{aligned}
\frac{1}{f(x)}f'(x) &= 1 + \frac{10}{x+\sin(x)}(1+\cos(x)) + \frac{13}{\ln(x)+\pi}\left(\frac{1}{x}\right) - \frac{1}{\arcsin(x)}\left(\frac{1}{\sqrt{1-x^2}}\right) - \frac{1}{\tan(x)}(\sec^2(x)) - \ln(2) \\
f'(x) &= \left(1 + \frac{10(1+\cos(x))}{x+\sin(x)} + \frac{13}{x(\ln(x)+\pi)} - \frac{1}{\arcsin(x)\sqrt{1-x^2}} - \frac{\sec^2(x)}{\tan(x)} - \ln(2)\right)f(x) \\
\therefore\ f'(x) &= \left(1 + \frac{10(1+\cos(x))}{x+\sin(x)} + \frac{13}{x(\ln(x)+\pi)} - \frac{1}{\arcsin(x)\sqrt{1-x^2}} - \frac{\sec^2(x)}{\tan(x)} - \ln(2)\right)\left(\frac{e^x(x+\sin(x))^{10}(\ln(x)+\pi)^{13}}{\arcsin(x)\tan(x)2^x}\right)
\end{aligned}
$$

Definition: Order of Derivatives

We can derive a derivative to get what is known as a second derivative. We can continue to derive (as desired) to get higher order derivatives. The order of a derivative is the number of times the variable has been derived.

y' = first order

y'' = second order

y''' = third order

Note that it would be cumbersome to continue to add more primes, thus we use parentheses to indicate "higher orders"

$y^{(n)}$ = nth order derivative

Example 6:

Determine the second derivative of the following function: $y = e^{-x^2}$

Solution:

We derive once using the chain rule to get:

$$y' = e^{\square}\square'$$

$$y' = e^{-x^2}(-2x)$$

$$= -2xe^{-x^2}$$

We then derive again to get:

$$y'' = A'B + AB'$$

$$= -2e^{-x^2} + (-2x)\left(-2xe^{-x^2}\right)$$

$$= -2e^{-x^2} + 4x^2e^{-x^2}$$

Examples: Order of Derivatives

Example 7:

Use implicit differentiation to find the value of y'' at the point $(0,1)$ given the following relation: $xy + 3e^x - \ln(y) = 3$

Solution:

We derive each side implicitly to get:

$$[xy]' + [3e^x]' - [\ln(y)]' = [3]'$$

$$y + xy' + 3e^x - \frac{y'}{y} = 0$$

Now we derive again to get us an expression with y'':

$$[y]' + [xy']' + [3e^x]' - \left[\frac{y'}{y}\right]' = [0]'$$

$$y' + y' + xy'' + 3e^x - \frac{(y''y - y'y')}{y^2} = 0$$

Now we see that to find the value of y'', we will need x, y, and y'. Thus we use our first derivative expression to find y' first:

$$y + xy' + 3e^x - \frac{y'}{y} = 0$$

$$1 + (0)y' + 3e^0 - \frac{y'}{1} = 0$$

$$4 - y' = 0$$

$$y' = 4$$

Then we can sub all of this into our second derivative expression to find our second derivative:

$$y' + y' + xy'' + 3e^x - \frac{(y''y - y'y')}{y^2} = 0$$

$$4 + 4 + (0)y'' + 3e^0 - \frac{(y''(1) - (4)(4))}{1^2} = 0$$

$$11 - y'' + 16 = 0$$

$$y'' = 27$$

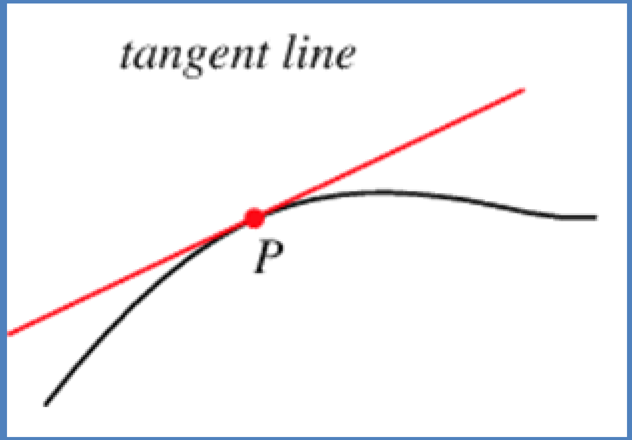
Thus the value of the second derivative at (0,1) is 27.

Formula: Linearization

The linearization at a point (a, f(a)) is simply another way to say "The equation of the tangent line". However, we can find a formula for the equation of the tangent line called the linearization formula:

Formula(s):	**When To Use it:**	**Proof of Formula(s)**
If we have a function $f(x)$ and we want to find the equation of the tangent line (linearization) at x = a, we can use the formula: $$L(x) = f(a) + f'(a)[x - a]$$	When you want the equation of a tangent line at x = a.	We know the equation of the tangent line is: $y = mx + b$ where $m = f'(a)$. We also know the point $(a, f(a))$ is on the line so if we sub all of this into the equation, we can solve for $b = f(a) - f'(a)[a]$ Putting it all into $y = mx + b$ and factoring out $f'(a)$ we get our linearization formula.

Strategy: Linearization For Approximation

Strategy:	When To Use it:	Why it works
If we want to approximate an x-value on a function (call it c) without a calculator, we would: 1) Determine an "a" value that we can evaluate exactly that is closest to "c" 2) Find the linearization at x = a. (Determine $f(x)$, $f'(x)$, $f(a)$, and $f'(a)$ and sub all of this into $L(x) = f(a) + f'(a)[x - a]$ 3) Sub in "c" into the linearization.	If we want to approximate an x-value on a function (call it c) without a calculator.	We note that a tangent line's y value is "close" to the y value of the function as long as the x-value is close: 

Examples: Using Linearization to Find Approximate Values

Example 8:

Find the Linearization at $x = 9$ for $f(x) = \sqrt{x}$, then use the linearization to find the approximate value of $\sqrt{8}$

Solution:

We note that: $f(x) = x^{1/2}$, and $f'(x) = \frac{1}{2}x^{-1/2}$. This gives us $f(a) = f(9) = 3$ and $f'(a) = f'(9) = \frac{1}{2}9^{-1/2} = \frac{1}{6}$.

We put all of this into our linearization formula to get:

$$\begin{aligned} L(x) &= f(a) + f'(a)[x-a] \\ &= 3 + \frac{1}{6}[x-9] \end{aligned}$$

We can then use this formula to find the approximate value of $\sqrt{8}$ as 8 is close to 9. We simply sub in $c = 8$ into our linearization to get:

$$\begin{aligned} L(8) &= 3 + \frac{1}{6}[8-9] \\ &= 3 - \frac{1}{6} \\ &= \frac{17}{6} \end{aligned}$$

We note that $\frac{17}{6} \sim 2.833$ which is close to $2.828 \sim \sqrt{8}$

Examples: Using Linearization to Find Approximate Values

Example 9:

Determine the most appropriate "a" value to approximate sin(1) and use this value (with its linearization) to determine the approximate value of sin(1)

Solution:

We know our function is $f(x) = \sin(x)$ which has a derivative of $f'(x) = \cos(x)$.

You may think that $a = 0$ is a good approximation, but we can actually do better!

We note that $\frac{\pi}{3}$ is a value that is almost 1 and is something we can evaluate exactly into the sin and cos functions! So we will use this instead.

$a = \frac{\pi}{3}$ $\quad \rightarrow f(a) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ $\quad$ and $\rightarrow f'(a) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$$\therefore L(x) = f(a) + f'(a)[x - a]$$

$$\rightarrow L(x) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right)$$

$$\rightarrow \mathrm{L}\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(1 - \frac{\pi}{3}\right)$$

$$= \frac{\sqrt{3}+1}{2} - \frac{\pi}{6}$$

Since we know $\pi \sim 3.14$ and $\sqrt{3} \sim 1.73$ we could use this to (carefully) evaluate this to become 0.842. If we actually sub in 1 into the sin function we get $\sin(1) = 0.841$. These values are very close!